Europ. J. Combinatorics (1989) **10**, 137–148

On Distance-regular Graphs with Fixed Valency, IV

EIICHI BANNAI AND TATSURO ITO

Dedicated to Professor Nagayoshi Iwahori on his 60th birthday

Distance-regular graphs with certain specific intersection arrays are investigated. In particular, it is shown that if the columns of the intersection array are all of the form ${}^t(*, 0, k)$, ${}^t(1, a, k - a - 1)$, ${}^t(k - a - 1, a, 1)$ and ${}^t(k - e, e, *)$ except for t intermediate columns, then the diameter d of the graph is bounded by a function depending only on t and the valency k if $k \geqslant 3$.

1. INTRODUCTION

This is the fourth of our sequels [2] which eventually intend to show that there are only finitely many distance-regular graphs with a fixed valency k, or equivalently, that the diameters of distance-regular graphs are bounded by a certain function of the valency k ($k \geqslant 3$).

Let $\Gamma = (X, R)$ be a distance-regular graph, where X and R are the vertex and edge sets. $\Gamma_i(x)$ denotes the set of vertices of distance i from $x \in X$. By the definition of distance-regularity, Γ is connected and the following numbers (which are called the intersection numbers),

$$\begin{aligned} c_i &= |\Gamma_{i-1}(x) \cap \Gamma_1(y)|, \\ a_i &= |\Gamma_i(x) \cap \Gamma_1(y)|, \\ b_i &= |\Gamma_{i+1}(x) \cap \Gamma_1(y)|, \end{aligned}$$

are independent of the choice $x \in X$ and $y \in \Gamma_i(x)$, and with these numbers we form an array (which is called the intersection array),

$$b = \begin{Bmatrix} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_{d-1} & * \end{Bmatrix}, \qquad (1)$$

where

$$k = b_0 = |\Gamma_1(x)|, \qquad d = \max\{i \mid \Gamma_i(x) \neq \varnothing\}$$

are the valency and the diameter of Γ. We often identify B with the tridiagonal matrix

$$\begin{bmatrix} 0 & 1 & & 0 & \\ k & a_1 & c_2 & & \\ & b_1 & a_2 & \ddots c_{d-1} & \\ & & b_2 & \ddots a_{d-1} & c_d \\ 0 & & & \ddots b_{d-1} & a_d \end{bmatrix}.$$

By the relations $b_i \geqslant b_{i+1}$, $c_i \leqslant c_{i+1}$ (cf. [1, p. 195]), we consider a distance-regular graph Γ whose intersection array B has $(1, a, k - a - 1)$ in the first r columns, $(k - a - 1, a, 1)$

0195-6698/89/020137+12 \$02.00/0

in the last s columns, and (c_i, a_i, b_i) in the intermediate t columns which are different from $(1, a, k - a - 1)$ and $(k - a - 1, a, 1)$:

$$B = \left\{ \begin{array}{ccccccccccccc} & \multicolumn{3}{c}{\overbrace{\hspace{8em}}^{r}} & \multicolumn{4}{c}{\overbrace{\hspace{8em}}^{t}} & \multicolumn{3}{c}{\overbrace{\hspace{10em}}^{s}} & \\ * & 1 & \cdots & 1 & c' & c'' & \cdots & c''' & k-a-1 & \cdots & k-a-1 & k-e \\ 0 & a & \cdots & a & a' & a'' & \cdots & a''' & a & \cdots & a & e \\ k & k-a-1 & \cdots & k-a-1 & b' & b'' & \cdots & b''' & 1 & \cdots & 1 & * \end{array} \right\}. \tag{2}$$

In this paper we shall show the following.

THEOREM 1. *Let Γ be a distance-regular graph of valency k ($k \geqslant 3$) and diameter d. Let t be the number of intermediate columns of the intersection matrix as shown above. Then d is bounded by a certain function f depending only on k and t, i.e. $d < f(k, t)$.*

In the subsequent papers, we want to bound t and hence d by a certain function of k.

Let θ be an eigenvalue of the adjacency matrix A of Γ and $m(\theta)$ the multiplicity of θ in A. As is well known, θ is an eigenvalue of B and $m(\theta)$ can be calculated from B in theory (cf. [1, p. 202]). To prove Theorem 1, we prepare a formula for $m(\theta)$ in which we separate the effects of the first r, intermediate t, last s columns of B in (2).

For an indeterminate x and given k, a in (2), let us define λ, μ, σ, τ by

$$\lambda + \mu = x - a, \qquad \lambda\mu = k - a - 1, \tag{3}$$

$$\sigma = \lambda/\mu, \qquad \tau = \mu/\lambda. \tag{4}$$

For B in (1), let us set

$$T_i = \begin{bmatrix} 0 & -b_{i-1}c_{i-1} \\ 1 & x - k + b_{i-1} + c_i \end{bmatrix} \quad (2 \times 2 \text{ matrix}). \tag{5}$$

Specializing B to be in the form (2), define the polynomials f_1, f_2, g_1, g_2 in x by

$$\begin{aligned} \begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} &= T_{r+1}T_{r+2} \cdots T_{r+t+1} \\ &= \begin{bmatrix} 0 & -k + a + 1 \\ 1 & x - a - 1 + c' \end{bmatrix} \begin{bmatrix} 0 & -b'c' \\ 1 & x - k + b' + c'' \end{bmatrix} \\ &\quad \cdots \begin{bmatrix} 0 & -b'''c''' \\ 1 & x - a - 1 + b''' \end{bmatrix}. \end{aligned} \tag{6}$$

Notice that f_1, f_2, g_1, g_2 are determined by the intermediate block of B illustrated in (2). Let us define P, Q, R, D as follows, where the prime stands for the derivative by x:

$$\begin{aligned} P &= (g_1 - \lambda f_1)/\mu + (g_2 - \lambda f_2), \\ Q &= (g_1 - \mu f_1)/\mu + (g_2 - \mu f_2), \end{aligned} \tag{7}$$

$$R = \frac{\mu + a + 1 - e}{\lambda + a + 1} P - \frac{\lambda + a + 1 - e}{\lambda + a + 1} Q\sigma^s, \tag{8}$$

$$D = \left(\frac{\mu + a + 1 - e}{\lambda + a + 1} P\right)' - \left(\frac{\lambda + a + 1 - e}{\lambda + a + 1} Q\right)' \sigma^s. \tag{9}$$

Notice again that P, Q are determined by the intermediate block of B illustrated in (2).

THEOREM 2. *Let Γ be a distance-regular graph whose intersection array is* (2). *let θ be an eigenvalue of Γ which is not equal to k or $-k/(a+1)$. Then the multiplicity $m(\theta)$ of θ in the adjacency matrix of Γ is given by*

$$m(\theta) = \frac{nk}{2}\frac{4(k-a-1)-(\theta-a)^2}{(k-\theta)(k+(a+1)\theta)}\frac{1}{M(\theta)},$$

where n is the number of vertices of Γ, and

$$M(\theta) = r + s\frac{(\lambda+a+1-e)(\mu+a+1-e)}{(\lambda+a+1)(\mu+a+1)}\frac{P\bar{P}-Q\bar{Q}}{R\bar{R}} + \frac{\lambda-\mu}{2}\frac{R\bar{D}-\bar{R}D}{R\bar{R}}, \tag{10}$$

P, Q, R, D being as before with $x = \theta$ and $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{D}$ being obtained from P, Q, R, D by interchanging λ, μ (and hence interchanging σ, τ). Furthermore, it holds that

$$P\bar{P} - Q\bar{Q} = -(\lambda-\mu)^2 b'c'b''c'' \cdots b'''c''' \qquad \textit{for any } x, \tag{11}$$

and

$$\bar{R}\sigma^{r+s} + R = 0 \qquad \textit{for} \qquad x = \theta. \tag{12}$$

The expression for $m(\theta)$ in this theorem seems technical but important, since it separates the effects on $m(\theta)$ of the three blocks of the intersection matrix B given by (2).

2. PROOF OF THEOREM 2

(1) The characteristic polynomial $(x-k)F_d(x)$ of B in (2) is given by

$$(F_{d-1}, F_d) = (1, x+1)T^{r-1}\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix}T^{s-1}(T+U), \tag{13}$$

where

$$t = \begin{bmatrix} 0 & -k+a+1 \\ 1 & x-a \end{bmatrix} = \begin{bmatrix} 0 & -\lambda\mu \\ 1 & \lambda+\mu \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & 0 \\ 0 & a-e+1 \end{bmatrix}, \tag{14}$$

and f_1, f_2, g_1, g_2 are the polynomials in x of degree $t-1, t, t, t+1$ which are defined by (6) (cf. [1, p. 202]). Notice that (13) holds for $s = 0$ as well, since

$$T^{-1}(T+U) = \begin{bmatrix} 0 & -b'''c''' \\ 1 & x-a-1+b''' \end{bmatrix}^{-1}\begin{bmatrix} 0 & -b'''c''' \\ 1 & x-e+b''' \end{bmatrix} = \begin{bmatrix} 1 & a+1-e \\ 0 & 1 \end{bmatrix}.$$

We want to calculate F_{d-1}, F_d in (13) and the derivative F_d' so that we can apply the formula

$$m(\theta) = \frac{nk(k-a-1)^{r+s}b'b''\cdots b'''c'c''\cdots c'''}{(k-\theta)F_{d-1}(\theta)F_d'(\theta)} \tag{16}$$

(cf. [1, p. 202]).

Let us set

$$E_i = \frac{\lambda^i-\mu^i}{\lambda-\mu}, \qquad E_i^* = (a+1)E_i + E_{i+1}. \tag{17}$$

Since λ, μ are eigenvalues of T, we have

$$(E_i, E_{i+1}) = (0, 1)T^i,$$

and since $(1, x+1) = (0, a+1) + (0, 1)T$, we have by (13)

$$(F_{d-1}, F_d) = (E^*_{r-1}, E^*_r)\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} T^{s-1}(T+U).$$

Let us set

$$\begin{aligned} f_r^* &= f_1 E^*_{r-1} + f_2 E^*_r, \\ g_r^* &= \{g_1 - (x-a)f_1\}E^*_{r-1} + \{g_2 - (x-a)f_2\}E^*_r. \end{aligned} \tag{18}$$

Then

$$(E^*_{r-1}, E^*_r)\begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} = f_r^*(1, x-a) + g_r^*(0, 1),$$

and so

$$(F_{d-1}, F_d) = \{f_r^*(E_s, E_{s+1}) + g_r^*(E_{s-1}, E_s)\}(T+U).$$

Thus we obtain

$$F_{d-1} = f_r^* E_{s+1} + g_r^* E_s \tag{19}$$

and

$$F_d = f_r^* E_{s+2} + g_r^* E_{s+1} + (a - e + 1)\{f_r^* E_{s+1} + g_r^* E_s\}. \tag{20}$$

(2) Let σ, τ be as in (4). Then it follows that

$$\begin{aligned} (\lambda-\mu)E_i &= \mu^i(\sigma^i - 1), \\ (\lambda-\mu)E_i^* &= \mu^i\{(a+1+\lambda)\sigma^i - (a+1+\mu)\}, \\ (\lambda-\mu)f_r^* &= \mu^r\left\{(a+1+\lambda)\left(\frac{f_1}{\lambda} + f_2\right)\sigma^r - (a+1+\mu)\left(\frac{f_1}{\mu} + f_2\right)\right\}, \\ (\lambda-\mu)g_r^* &= \mu^r\left\{(a+1+\lambda)\left(\frac{g_1 - (x-a)f_1}{\lambda} + g_2 - (x-a)f_2\right)\sigma^r \right. \\ &\qquad \left. - (a+1+\mu)\left(\frac{g_1 - (x-a)f_1}{\mu} + g_2 - (x-a)f_2\right)\right\}. \end{aligned}$$

Since $x - a = \lambda + \mu$, we obtain by (19)

$$\begin{aligned} \frac{(\lambda-\mu)^2}{\mu^{r+s}} F_{d-1} &= (a+1+\lambda)\sigma^r\left\{\left(\frac{f_1}{\sigma} + \mu f_2\right)(\sigma^{s+1} - 1) \right. \\ &\qquad \left. + \left(\frac{g_1 - (x-a)f_1}{\lambda} + g_2 - (x-a)f_2\right)\left(\sigma^s - 1\right)\right\} \\ &\quad - (a+1+\mu)\left\{(f_1 + \mu f_2)(\sigma^{s+1} - 1) \right. \\ &\qquad \left. + \left(\frac{g_1 - (x-a)f_1}{\mu} + g_2 - (x-a)f_2\right)(\sigma^s - 1)\right\} \\ &= (a+1+\lambda)\sigma^r\left\{\sigma^s\left(\frac{g_1 - \mu f_1}{\lambda} + g_2 - \mu f_2\right) - \left(\frac{g_1 - \lambda f_1}{\lambda} + g_2 - \lambda f_2\right)\right\} \\ &\quad - (a+1+\mu)\left\{\sigma^s\left(\frac{g_1 - \mu f_1}{\mu} + g_2 - \mu f_2\right) \right. \\ &\quad \left. - \left(\frac{g_1 - \lambda f_1}{\mu} + g_2 - \lambda f_2\right)\right\}. \end{aligned}$$

Thus we have

$$\frac{(\lambda - \mu)^2}{\mu^{r+s}} F_{d-1} = (a + 1 + \lambda)\sigma^r(\sigma^s \bar{P} - \bar{Q}) - (a + 1 + \mu)(\sigma^2 Q - P), \tag{21}$$

where P, Q are as in (7) and $\bar{P}$, $\bar{Q}$ are obtained from P, Q by interchanging λ, μ.

By (20) we have

$$\frac{(\lambda - \mu)^2}{\mu^{r+s}} F_d = (a + 1 + \lambda)\sigma^r\{\sigma^s(\lambda + a - e + 1)\bar{P} - (\mu + a - e + 1)\bar{Q}\}$$
$$- (a + 1 + \mu)\{\sigma^s(\lambda + a - e + 1)Q - (\mu + a - e + 1)P\},$$

so

$$\frac{(\lambda - \mu)^2}{(\lambda + a + 1)(\mu + a + 1)\mu^{r+s}} F_d = \sigma^r\left(\sigma^s \frac{\lambda + a + 1 - e}{\mu + a + 1} \bar{P} - \frac{\mu + a + 1 - e}{\mu + a + 1} \bar{Q}\right)$$
$$- \left(\sigma^s \frac{\lambda + a + 1 - e}{\lambda + a + 1} Q - \frac{\mu + a + 1 - e}{\lambda + a + 1} P\right) \tag{22}$$

$$= \sigma^{r+s}\bar{R} + R, \tag{23}$$

where R is as in (8) and $\bar{R}$ is obtained from R by interchanging λ, μ (and hence by interchanging σ, τ). Thus we obtain the identity (12).

In what follows, we assume θ is an eigenvalue of Γ ($\theta \neq k$), i.e. $F_d(\theta) = 0$. We also assume $\theta \neq -k/(a + 1)$, i.e. $(\lambda + a + 1)(\mu + a + 1) \neq 0$.

(3) By (21) and (22) we have

$$\frac{(\lambda - \mu)^2}{\mu^{r+s}}\left(F_{d-1} - \frac{1}{\mu + a + 1} F_d\right) = (\lambda + a + 1)\sigma^r$$

$$\left\{\frac{\mu - \lambda + e}{\mu + a + 1} \bar{P}\sigma^s - \frac{e}{\mu + a + 1} \bar{Q}\right\} - (\mu + a + 1)\left\{\frac{u - \lambda + e}{\mu + a + 1} Q\sigma^s - \frac{e}{\mu + a + 1} P\right\}.$$

Since $F_d(\theta) = 0$, we have by (22)

$$\frac{\lambda + a + 1}{\mu + a + 1}\sigma^r = \frac{(\lambda + a + 1 - e)Q\sigma^s - (\mu + a + 1 - e)P}{(\lambda + a + 1 - e)\bar{P}\sigma^s - (\mu + a + 1 - e)\bar{Q}}. \tag{24}$$

Hence

$$\frac{(\lambda - \mu)^2}{\mu^{s+s}}\left(F_{d-1} - \frac{1}{\mu + a + 1} F_d\right) = \frac{(\lambda + a + 1 - e)Q\sigma^s - (\mu + a + 1 - e)P}{(\lambda + a + 1 - e)\bar{P}\sigma^s - (\mu + a + 1 - e)\bar{Q}}$$
$$\times \{(\mu - \lambda + e)\bar{P}\sigma^s - e\bar{Q}\}$$
$$- \{(\mu - \lambda + e)Q\sigma^s - eP\}$$
$$= \frac{(\lambda - \mu)(\mu + a + 1)(P\bar{P} - Q\bar{Q})\sigma^s}{(\lambda + a + 1 - e)\bar{P}\sigma^s - (\mu + a + 1 - e)\bar{Q}}$$
$$= \frac{(\lambda - \mu)(\lambda + a + 1)(P\bar{P} - Q\bar{Q})}{(\lambda + a + 1 - e)Q\sigma^s - (\mu + a + 1 - e)P}\sigma^{r+s}.$$

Thus we obtain

$$\frac{\lambda - \mu}{\lambda^{r+s}}\left(F_{d-1} - \frac{1}{\mu + a + 1} F_d\right) = \frac{P\bar{P} - Q\bar{Q}}{R},$$

and so

$$\frac{\lambda - \mu}{\lambda^{r+s}} F_{d-1} = -\frac{P\bar{P} - Q\bar{Q}}{R}. \tag{25}$$

It is easy to check

$$\begin{bmatrix} \bar{P} & \bar{Q} \\ Q & P \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1 \\ 1/\mu & 1 \end{bmatrix} \begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} \begin{bmatrix} -\mu & -\lambda \\ 1 & 1 \end{bmatrix}. \tag{26}$$

Taking the determinant of (26) and (6), we obtain

$$\begin{aligned} P\bar{P} - q\bar{Q} &= -\frac{(\lambda - \mu)^2}{\lambda\mu}(k - a - 1)b'c'b''c'' \cdots b'''c''' \\ &= -(\lambda - \mu)^2 b'c'b''c'' \cdots b'''c''', \end{aligned}$$

which is the identity (11).

(4) The derivatives of λ, μ, σ, τ by x are

$$\lambda' = \frac{\lambda}{\lambda - \mu}, \quad \mu' = -\frac{\mu}{\lambda - \mu}, \quad \sigma' = \frac{2\sigma}{\lambda - \mu}, \quad \tau' = -\frac{2\tau}{\lambda - \mu}. \tag{27}$$

The operation of taking derivatives (indicated by a prime) commutes with the operation shown by a bar sign of interchanging λ, μ. Since $F_d(\theta) = 0$, the derivative of (22) at $x = \theta$ is

$$\begin{aligned} \frac{(\lambda - \mu)^2}{(\lambda + a + 1)(\mu + a + 1)\mu^{r+s}} F_d' &= \frac{2r}{\lambda - \mu}\sigma^r \left(\sigma^s \frac{\lambda + a + 1 - e}{\mu + \alpha + 1}\bar{P}\right. \\ &\quad \left. - \frac{\mu + a + 1 - e}{\mu + a + 1}\bar{Q}\right) + \sigma^r \left\{\frac{2s}{\lambda - \mu}\sigma^s \frac{\lambda + a + 1 - e}{\mu + a + 1}\bar{P}\right. \\ &\quad \left. + \sigma^s \left(\frac{\lambda + a + 1 - e}{\mu + a + 1}\bar{P}\right)' - \left(\frac{\mu + a + 1 - e}{\mu + a + 1}\bar{Q}\right)'\right\} \\ &\quad - \left\{\frac{2s}{\lambda - \mu}\sigma^s \frac{\lambda + a + 1 - e}{\lambda + a + 1}\dot{Q}\right. \\ &\quad \left. + \sigma^s \left(\frac{\lambda + a + 1 - e}{\lambda + a + 1}Q\right)' - \left(\frac{\mu + a + 1 - e}{\lambda + a + 1}P\right)'\right\} \\ &= \frac{2r}{\lambda - \mu}\sigma^{r+s}\bar{R} + \frac{2s}{\lambda - \mu}\sigma^s \left(\sigma^r \frac{\lambda + a + 1 - e}{\mu + a + 1}\bar{P}\right. \\ &\quad \left. - \frac{\lambda + a + 1 - e}{\lambda + a - 1}Q\right) + \sigma^{r+s}\bar{D} + D, \end{aligned}$$

where D is as in (9) and $\bar{D}$ is obtained from D by interchanging λ, μ (and hence interchanging σ, τ).

By (23), we have

$$\sigma^{r+s}\bar{R} = -R.$$

So

$$\sigma^s\left(\sigma^r \frac{\lambda + a + 1 - e}{\mu + a + 1} \bar{P} - \frac{\lambda + a + 1 - e}{\lambda + a + 1} Q\right)$$

$$= -\frac{1}{\bar{R}}\left(\frac{\lambda + a + 1 - e}{\mu + a + 1} \bar{P}R + \frac{\lambda + a + 1 - e}{\lambda + a + 1} Q\bar{R}\sigma^s\right)$$

$$= -\frac{(\lambda + a + 1 - e)(\mu + a + 1 - e)}{(\lambda + a + 1)(\mu + a + 1)} \frac{P\bar{P} - Q\bar{Q}}{\bar{R}}. \tag{28}$$

Therefore,

$$\frac{(\lambda - \mu)^2}{(\lambda + a + 1)(\mu + a + 1)\mu^{r+s}} F_d' = -\frac{2r}{\lambda - \mu} R - \frac{2s}{\lambda - \mu}$$

$$\times \frac{(\lambda + a + 1 - e)(\mu + a + 1 - e)}{(\lambda + a + 1)(\mu + a + 1)} \frac{P\bar{P} - Q\bar{Q}}{\bar{R}} - \frac{R\bar{D} - \bar{R}D}{\bar{R}} = -\frac{2}{\lambda - \mu} RM(\theta), \tag{29}$$

where $M(\theta)$ is as in (10).

By (25) and (29),

$$F_{d-1}F_d' = \frac{2(\lambda + a + 1)(\mu + a + 1)(\lambda\mu)^{r+s}}{(\lambda - \mu)^4} (P\bar{P} - Q\bar{Q})M(\theta)$$

$$= -\frac{2(\lambda + a + 1)(\mu + a + 1)}{(\lambda - \mu)^2} (\lambda\mu)^{r+s} b'c'b''c'' \cdots b'''c'''M(\theta).$$

By (16), we obtain Theorem 2, as

$$(\lambda + a + 1)(\mu + a + 1) = k + (a + 1)\theta, \quad (\lambda - \mu)^2 = (\theta - a)^2 - 4(k - a - 1).$$

REMARK. We observe that

$$\begin{bmatrix} \bar{P} & \bar{Q} \\ Q & P \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1 \\ 1/\mu & 1 \end{bmatrix} \begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} \begin{bmatrix} -\mu & -\lambda \\ 1 & 1 \end{bmatrix},$$

$$T = \frac{1}{\lambda - \mu} \begin{bmatrix} -\mu & -\lambda \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda^2 & 0 \\ 0 & -\mu^2 \end{bmatrix} \begin{bmatrix} 1/\lambda & 1 \\ 1/\mu & 1 \end{bmatrix},$$

$$(\lambda - \mu) \begin{bmatrix} 1/\lambda & 0 \\ 0 & -1/\mu \end{bmatrix} = \begin{bmatrix} 1/\lambda & 1 \\ 1/\mu & 1 \end{bmatrix} \begin{bmatrix} -\mu & -\lambda \\ 1 & 1 \end{bmatrix},$$

and these relations lead to (21) and (22). For the same reason, if one defines K, L by

$$\begin{bmatrix} -\bar{K} & -\bar{L} \\ L & K \end{bmatrix} = \begin{bmatrix} 1 & \lambda \\ 1 & \mu \end{bmatrix} \begin{bmatrix} f_1 & g_1 \\ f_2 & g_2 \end{bmatrix} \begin{bmatrix} \mu & -\lambda \\ -1 & 1 \end{bmatrix},$$

then one may express $m(\theta)$ in terms of K, L because of the relations

$$T = \frac{1}{\mu - \lambda} \begin{bmatrix} \mu & -\lambda \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 1 & \mu \end{bmatrix},$$

$$(\mu - \lambda) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mu & -\lambda \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda \\ 1 & \mu \end{bmatrix}.$$

But, as is easily checked, $K = \mu P$ and $L = -\mu Q$. We wonder whether there are any other expressions of $m(\theta)$ which separate the effects of the three blocks of the intersection matrix B as illustrated in (2). Perhaps it is difficult (or unlikely to find other such expressions.

3. Proof of Theorem 1

(1) Let $(x - k)F_r(x)$ be the minimal polynomial of the $(r + 1) \times (r + 1)$ tridiagonal matrix

$$\begin{Bmatrix} * & 1 & \cdots & 1 & 1 \\ 0 & a & \cdots & a & k-1 \\ k & k-a-1 & \cdots & k-a-1 & * \end{Bmatrix}, \tag{30}$$

which is obtained by picking up the $(r + 1) \times (r + 1)$ top square of B (see (2)) and changing the $(r + 1, r + 1)$ entry to $k - 1$ to make the column sum k. Let $(x - k)F_d(x)$ be the minimal polynomial of B. The roots of $F_d(x)$ separate those of $F_r(x)$, as there exist orthogonal polynomials $\{F_i(x)\}_{0 \leqslant i \leqslant d}$ (cf. [1, p. 202]). F_r has a root in each open interval

$$\left(a + 2\sqrt{k-a-1}\cos\frac{i\pi}{r},\ a + 2\sqrt{k-a-1}\cos\frac{i\pi}{r+1}\right) \tag{31}$$

for $1 \leqslant i \leqslant r$, since $F_r = E_r + E_{r+1}$ has opposite signs at the end points of the interval, where E_i's are as in (17). In particular, F_d has a root θ such that

$$\theta = a + 2\sqrt{k-a-1}\cos\phi, \quad \frac{\pi}{r+1} < \phi < \frac{2\pi}{r}, \tag{32}$$

and the corresponding λ, μ in (3) are

$$\lambda = \sqrt{k-a-1}\, e^{\sqrt{-1}\phi}, \qquad \mu = \sqrt{k-a-1}\, e^{-\sqrt{-1}\phi}. \tag{33}$$

(2) We want to show the following:

Proposition 3. *$F_d(x)$ has at least $r + s - t - 4$ roots in the open interval*

$$(a - 2\sqrt{k-a-1},\ \ a + 2\sqrt{k-a-1}). \tag{34}$$

First, we symmetrize B in (1). Let

$$\Delta = \operatorname{diag}(1, \sqrt{k}, \sqrt{k_2}, \ldots, \sqrt{k_d}), \tag{35}$$

where k_i is the ith valency

$$k_i = b_0 b_1 \cdots b_{i-1}/c_1 c_2 \cdots c_i. \tag{36}$$

Then we obtain a symmetric matrix

$$\Delta^{-1}B\Delta = \begin{Bmatrix} * & \cdots & \sqrt{b_{i-1}c_i} & \cdots & \sqrt{b_{d-1}c_d} \\ 0 & \cdots & a_i & \cdots & a_d \\ \sqrt{k} & \cdots & \sqrt{b_i c_{i+1}} & \cdots & * \end{Bmatrix}. \tag{37}$$

The following is well known (cf. [6, p. 8]).

Theorem. *Let S be a $(p \times p)$ symmetric matrix over $\mathbb{R}$. Let S_i be the matrix obtained by dropping the ith row and the ith column from S. Let $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_p$ be the eigenvalues of S and $\beta_1 \geqslant \beta_2 \geqslant \cdots \geqslant \beta_{p-1}$ the eigenvalues of S_i, allowing repetition. Then*

$$\alpha_1 \geqslant \beta_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_j \geqslant \beta_j \geqslant \cdots \geqslant \beta_{p-1} \geqslant \alpha_p.$$

Now, specializing B as in (2), from the symmetric matrix $\Delta^{-1}B\Delta$, we drop the ith row and column for $r \leqslant i \leqslant r + t$ and $i = d$ successively to obtain

$$B' = \begin{array}{cc} \begin{array}{cc} r & s \end{array} & \\ \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} & \begin{array}{c} r \\ s \end{array} \end{array},$$

where

$$B_1 = \left\{\begin{array}{cccccc} * & \sqrt{k} & \sqrt{k-a-1} & \cdots & \sqrt{k-a-1} & \sqrt{k-a-1} \\ 0 & a & a & \cdots & a & a \\ \sqrt{k} & \sqrt{k-a-1} & \sqrt{k-a-1} & \cdots & \sqrt{k-a-1} & * \end{array}\right\}$$

and

$$B_2 = \left\{\begin{array}{ccccc} * & \sqrt{k-a-1} & \cdots & \sqrt{k-a-1} & \sqrt{k-a-1} \\ a & a & \cdots & a & a \\ \sqrt{k-a-1} & \sqrt{k-a-1} & \cdots & \sqrt{k-a-1} & * \end{array}\right\}$$

B_1 has at least $r - 1$ eigenvalues in the interval (34), since B_1 is obtained by dropping the last row and column of the symmetrization of the $(r + 1) \times (r + 1)$ tridiagonal matrix (30), which has r eigenvalues in the same interval. B_2 has at least $s - 1$ eigenvalues in the same interval, since if we drop the first row and column from B_1 we obtain a matrix like B_2 of size $r - 1$ which has at least $r - 2$ eigenvalues in the same interval. Hence B' has at least $r + s - 2$ eigenvalues in the same interval. Each time we drop the ith row and column from B to obtain B', the number of eigenvalues in the interval (34) increases at most by one. Since we drop the rows and columns $(t + 2)$ times from B, B' has at most $t + 2$ more eigenvalues than B in the interval (34). Therefore, B has at least $r + s - t - 4$ eigenvalues in the same interval.

(3) Let us fix k, t and let r tend to infinity. Choose an eigenvalue θ as in (32) and consider the product

$$\Pi\{4(k - a - 1) - (\theta' - a)^2\}, \tag{38}$$

where the product is over all θ' which are algebraic conjugates of θ over the rational number field. The product is a non-zero integer.

By Proposition 3, there are at most $2t + 5$ algebraic conjugates of θ outside the interval (34). Therefore the contribution of such θ' to the product (38) is at most

$$(4k^2)^{2t+5} \tag{39}$$

in the absolute value, as all the eigenvalues of B are in $[-k, k]$ by the Perron–Frobenius theorem. By (32), the contribution of θ to the product is

$$4(k - a - 1)(1 - \cos^2\phi). \tag{40}$$

The product of (39) and (40) is less than 1, if r is large enough compared with k, t. Hence there exists an algebraic conjugate θ' of θ in the interval (34) whose contribution to the product (38) is greater than 1, i.e.

$$\theta' \in (a - \sqrt{4k - 4a - 5},\ \ a + \sqrt{4k - 4a - 5}). \tag{41}$$

We want to derive a contradiction by showing that

$$m(\theta) < m(\theta'), \tag{42}$$

because it must hold that $m(\theta) = m(\theta')$ for algebraic conjugate eigenvalues θ, θ'.

(4) We use Theorem 2 to show

$$C_1 \frac{n}{r} < m(\theta') < C_2 \frac{n}{r} \tag{43}$$

for some positive constants C_1, C_2 depending only on k, t.

Let λ, μ be as in (3) for $x = \theta'$. Then λ, μ are not real numbers and complex conjugates of each other. So the operation shown by a bar sign is complex conjugation. Since the imaginary parts of λ, μ have absolute values greater than some positive constant depending only on k, the absolute values of $\lambda + a + 1, \lambda + a + 1 - e$ are bounded from above and below by some positive constants depending only on k. Also, the absolute values of P, Q, R, D are bounded from above, as k, t are fixed.

Since λ, μ are complex conjugates, we have $|\lambda + a + 1 - e| = |\mu + a + 1 - e|$, $|\sigma| = 1$ and so

$$|R| \geqslant \frac{|\lambda + a + 1 - e|}{|\lambda + a + 1|} (|P| - |Q|). \tag{44}$$

By (11),

$$\begin{aligned} |P| - |Q| &= (P\bar{P} - Q\bar{Q})/(|P| + |Q|) \\ &= -(\lambda - \mu)^2 b'c'b''c'' \cdots b'''c'''/(|P| + |Q|), \end{aligned} \tag{45}$$

where $-(\lambda - \mu)^2 = 4(k - a - 1) - (\theta' - a)^2$. Therefore, $|R|$ is greater than some positive constant.

With all these bounds for $M(\theta')$ in (10), we have

$$M(\theta') = r + K_1 s + K_2 \qquad (K_1 > 0),$$

where K_1, $|K_2|$ are bounded from above by some constant depending only on k, t. We want to show

$$s \leqslant r. \tag{46}$$

Suppose $s \geqslant r + 1$. Then by $c_i \leqslant b_j$ for $i + j \geqslant d$ (cf. [1, p. 244]), we have $b_{r+1} \geqslant c_{d-r-1} = k - a - 1$ and $c_{r+1} \leqslant b_{d-r-1} = 1$. By $b_i \geqslant b_{i+1}$, $c_i \leqslant c_{i+1}$ (cf. [1, p. 195]), we have $b_{r+1} \leqslant b_r = k - a - 1$ and $c_{r+1} \geqslant c_r = 1$. So $b_{r+1} = k - a - 1$, $c_{r+1} = 1$, which contradicts the definition of r. Thus we obtain (46) and hence (43).

(5) We want to show that

$$m(\theta) < C_3 n/r^3 \tag{47}$$

for some constant C_3 depending only on k. We use the following multiplicity formula (cf. [1, pp. 72, 190])

$$m(\theta) = n \Big/ \sum_{i=0}^{d} (v_i(\theta)^2/k_i), \tag{48}$$

where $v_i(x)$ $(0 \leqslant i \leqslant d)$ are polynomials defined by the recurrence

$$xv_i = b_{i-1}v_{i-1} + a_i v_i + c_{i+1}v_{i+1},$$

with $v_0(x) = 1$, $v_1(x) = x$, and k_i is the ith valency given by (36).

For B in (2), we have

$$xv_i = (k - a - 1)v_{i-1} + av_i + v_{i+1} \qquad (2 \leqslant i \leqslant r - 1), \tag{49}$$

with $v_1(x) = x$, $v_2(x) = x^2 - ax - k$, and

$$k_i = k(k - a - 1)^{i-1} \qquad (1 \leqslant i \leqslant r). \tag{50}$$

Let λ, μ be as in (33) and T as in (14). Then for $1 \leqslant i \leqslant r - 1$

$$(v_i, v_{i+1}) = (v_1, v_2)T^{i-1}.$$

Since $(v_1, v_2) = (0, 1)T^2 + a(0, 1)T - (a + 1)(0, 1)$, we have

$$v_i = E_{i+1} + aE_i - (a + 1)E_{i-1},$$

where the E_j's are as in (17). So, for $1 \leqslant i \leqslant r$,

$$v_i = \frac{1}{\lambda - \mu}\{(\lambda^2 + a\lambda - a - 1)\lambda^{i-1} - (\mu^2 + a\mu - a - 1)\mu^{i-1}\}, \tag{51}$$

$$v_i = \frac{\sqrt{k - a - 1}^{\,i-2}}{\sin\phi}\{(k - a - 1)\sin(i + 1)\phi + a\sqrt{k - a - 1}\sin i\phi$$
$$- (a + 1)\sin(i - 1\phi\}. \tag{52}$$

Since $\pi/(r + 1) < \phi < 2\pi/r$ in (32) and r is large enough, we have, by replacing $\sin(i \pm 1)\phi$ by $\sin i\phi$,

$$v_i = \frac{\sqrt{k - a - 1}^{\,i-2}}{\sin\phi}\{(k - a - 1 + a\sqrt{k - a - 1} - a - 1)\sin i\phi + \varepsilon\}, \tag{53}$$

with $|\varepsilon| < K/r$ for some constant K depending only on k.

Suppose $k - a - 1 \neq 1$. Then $k - a - 1 \geqslant 2$ and so

$$k - a - 1 + a\sqrt{k - a - 1} - a - 1 = (\sqrt{k - a - 1} + a + 1)(\sqrt{k - a - 1} - 1)$$

is greater than some positive constant. By (53), if r is large enough,

$$v_i > rL\sqrt{k - a - 1}^{\,i-2} \tag{54}$$

for $\delta < i\phi < \pi - \delta$, where δ is an absolute constant, say $\delta = \pi/10$, and L is a constant depending only on k. Hence, taking the sum in (48) over i with $\delta < i\phi < \pi - \delta$, say over i with $r/4 \leqslant i \leqslant r/3$, we have

$$\begin{aligned} m(\theta) &< n \Big/ \sum_{r/4 \leqslant i \leqslant r/3} (v_i(\theta)^2/k_i) \\ &< n/([r/3] - [r/4])(r^2L^2/k(k - a - 1)) \\ &< C_3 n/r^3 \end{aligned}$$

for some constant C_3 depending only on k.

Supose $k - a - 1 = 1$. Then by $b_i \geqslant b_{i+1}$, $c_i \leqslant c_{i+1}$, all b_i, c_i must be 1 and so $s = t = 0$ in (2). In particular, $k_1 = k_2$ and we obtain $d \leqslant 3$ by the theorem of Taylor-Levingston (cf. [1, p. 245]).

By (43) and (47) we obtain (42).

4. Remarks

(1) The classification of distance-regular graphs with valency 3 in [3] is mainly based on the method of counting circuit pattern changes by the use of the idea of [5]. But they are also classified by a purely algebraic method as follows. Let Γ be a distance-regular graph with valency 3. Then the number of (c_i, a_i, b_i) with $(c_i, a_i, b_i) = (1, 1, 1)$ is bounded by [2, I], i.e. t in Theorem 1 is bounded. Hence the diameter is bounded by Theorem 1, and also it is possible to complete the classification of cubic distance-regular graphs by this method.

(2) Theorem 1 also implies that there are only finitely many distance-regular graphs of valency 4. Let Γ be a distance-regular graph with valency 4. Then the number of (c_i, a_i, b_i) with $(c_i, a_i, b_i) = (1, 2, 1), (2, 0, 2)$ is bounded by [2, I]. So, if $(c_1, a_1, b_1) = (1, 1, 2)$, then t in Theorem 1 is bounded. If $(c_1, a_1, b_1) = (1, 0, 3)$, then the number of (c_i, a_i, b_i) with $(c_i, a_i, b_i) = (1, 1, 2), (2, 1, 1)$ is at most 8 by [4] (and by a similar argument). So in this case, t in Theorem 1 is also bounded. Hence the diameter is bounded by Theorem 1.

With Kazumasa Nomura, the authors are working on the complete classification of distance-regular graphs with valency 4.

Acknowledgements

E. Bannai was supported in part by NSF grant DMS-8503761. This work was done while T. Ito was visiting the Ohio State University in 1986/7.

References

1. E. Bannai and T. Ito, *Algebraic combinatorics I*, Benjamin/Cummings, Menlo Park, California, 1984.
2. E. Bannai and T. Ito. On distance-regular graphs with fixed valency I, II, III; I *Graphs and Combinatorics*, **3** (1987), 95–109; II *Graphs and Combinatorics* **4** (1988), 219–228; III *J. Algebra* **107** (1987), 43–52.
3. N. L. Biggs, A. G. Boshier and J. Shawe-Taylor, Cubic distance-regular graphs, *J. Lond. Math. Soc.* (2), **33** (1986), 385–394.
4. A. G. Boshier and K. Nomura, A remark on the intersection arrays of distance-regular graphs, to appear in *J. Combin. Theory* (*B*) **44** (1988), 147–153.
5. A. A. Ivanov, Bounding the diameter of a distance-regular graph, *Soviet Math. Dokl.*, **28** (1983), 149–152.
6. A. Schrijver, *Packing and covering in combinatorics*, Math. Center Tracts 106, Mathematisch Centrum, Amsterdam.

Received 30 March 1987 and in revised form 29 December 1987

Eiichi Bannai
Department of Mathematics, The Ohio State University, Columbus, Ohio 43210, U.S.A.
and
Tatsuro Ito
Joetsu University of Education, Joetsu, Niigata 943, Japan